

Topological Hopf algebras and their Hopf-cyclic cohomology

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Abstract

A natural extension of Hopf-cyclic cohomology is introduced to encompass topological Hopf algebras and topological coefficients. It is shown that the topological coefficients properly include the algebraic ones. The topological theory is more satisfactory than the algebraic one; to wit, contrary to the algebraic case, there is a one-to-one correspondence between the topological coefficients over a Lie algebra and those over its universal enveloping algebra equipped with the strict inductive limit topology. For topological Hopf algebras the category of topological coefficients is identified with the representation category of a topological algebra called the anti-Drinfeld double. This is a generalization of an existing such identification for finite dimensional Hopf algebras. A topological van Est type isomorphism is detailed, connecting the Hopf-cyclic cohomology to the relative Lie algebra cohomology with respect to a maximal compact subalgebra.

Contents

1	Introduction	2
2	Hopf-cyclic cohomology for topological Hopf algebras	3
2.1	Topological Hopf algebras	3
2.2	Topological Lie-Hopf algebras	6

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3	Hopf-cyclic cohomology for topological Hopf algebras	10
3.1	Hopf-cyclic complex for topological Hopf algebras . . .	10
3.2	Characterization of Hopf-cyclic coefficients	15
4	Cyclic cohomology for topological Lie algebras	17
4.1	Corepresentations of topological Lie algebras	17
4.2	Cyclic cohomology theories for topological Lie algebras	22
5	Computation	27
5.1	Coalgebra Hochschild cohomology of $\mathcal{F}^\infty(G)$	27
5.2	Hopf-cyclic cohomology of $\mathcal{F}^\infty(G_2) \blacktriangleright_{\pi} U(\mathfrak{g}_1)$	30

1 Introduction

Hopf-cyclic cohomology emerged naturally as a byproduct of the fundamental work of Connes-Moscovici on their local index formula [5]. Later on, the theory was furnished with a category of coefficients called the stable-anti-Yetter-Drinfeld (SAYD) modules [9, 10, 13]. The theory of coefficients has also an unexplored extension for bialgebras as well [14].

In [27, 28, 29] we investigated the category of SAYD modules over the classical and nonclassical Hopf algebras. We defined such coefficients for Lie algebras as an enlargement of the category of representations of Lie algebras. We showed that such coefficients, provided to be co-nilpotent, are in one to one correspondence with the SAYD modules over the enveloping algebra of the Lie algebra in question. In particular, for any natural number q , the space of $2q$ -truncated polynomials over a Lie algebra is naturally a SAYD module over the Lie algebra, and furthermore it can be exponentiated to a SAYD module over the enveloping algebra of the Lie algebra. However, the exponentiation procedure holds merely due to the nilpotency of the coaction in question. For an arbitrary comodule this procedure fails unless we endow the enveloping algebra and the coefficient space with a suitable topology.

Another occasion supporting a topological extension of the Hopf-cyclic theory is [21], in which one of the authors and H. Moscovici defined a Hopf algebra \mathcal{K}_n as the symmetry of the crossed product algebra $C_c^\infty(\mathbb{R}^n) \rtimes \text{Diff}(\mathbb{R}^n)$. The cohomology of \mathcal{K}_n surprisingly consists of only Chern classes, and misses the secondary characteristic classes of

foliations. This shortage will be fixed by replacing \mathcal{K}_n with a topological Hopf algebra \mathcal{G}_n in [22].

These incidents encouraged us to develop the current Hopf-cyclic cohomology theory for topological Hopf algebras. We strongly believe that our study will be followed up by the development of a suitable Chern-Weil theory for quantum groups.

Unless stated otherwise, throughout the text a vector space is assumed to be well-behaved, [1, 2], that is either nuclear and Fréchet, or nuclear and dual¹ of Fréchet, [8, 34]. We use the projective tensor product \otimes_π , and its completion $\widehat{\otimes}_\pi$ for the tensor product of topological vector spaces (t.v.s.). For details we refer the reader to [32, 34]. On the other hand, \mathfrak{g} denotes a countable dimensional Lie algebra, k denotes the ground field, and \otimes refers to the tensor product over k .

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2 Hopf-cyclic cohomology for topological Hopf algebras

In this section we define the Hopf-cyclic cohomology for topological Hopf algebras.

2.1 Topological Hopf algebras

Let us first recall the notion of a topological Hopf algebra from [1, Def. 1.2], see also [2, Def. 2], in which they are defined as well-behaved Hopf algebras.

Definition 2.1. *A Hopf algebra (resp. algebra, coalgebra) H whose underlying vector space is a t.v.s. is called a topological Hopf algebra (resp. algebra, coalgebra) if the Hopf algebra structure maps (resp. algebra, coalgebra structure maps) are continuous.*

Any countable dimensional Hopf algebra, equipped with the strict inductive limit topology [34, Sect. I.13], is a topological Hopf algebra by [1, Prop. 1.5.1]. We list below more specific examples.

¹Strong dual, [32, Sect. II.2.3]

Example 2.2. The universal enveloping algebra $U(\mathfrak{g})$ of a (countable dimensional) Lie algebra \mathfrak{g} .

Example 2.3. The Hopf algebra $R(G)$ of representative functions on a compact connected Lie group G , [1, Sect. 2.4], see also [2] for a linear or semi-simple Lie group G .

For a further example, the Hopf algebra of (infinitely) differentiable functions on a real analytic group, we adopt the terminology of [11].

Definition 2.4. Let G be a real analytic group, and let \mathfrak{g} be its Lie algebra. Let also V be a t.v.s. Then, a continuous map $\varphi : G \longrightarrow V$ is called differentiable if

(i) for any $g \in G$, $X \in \mathfrak{g}$, and $t \in \mathbb{R}$,

$$\varphi(g \exp(tX)) = \varphi(g) + t\tilde{\varphi}(g, X, t),$$

for a continuous map $\tilde{\varphi} : G \times \mathfrak{g} \times \mathbb{R} \longrightarrow V$,

(ii) for any $X \in \mathfrak{g}$, and differentiable $\varphi : G \longrightarrow V$,

$$X(\varphi) : G \longrightarrow V, \quad X(\varphi)(g) := \tilde{\varphi}(g, X, 0)$$

satisfies (i), as well as $Y(X(\varphi))$ for any $X, Y \in \mathfrak{g}$.

Example 2.5. Let G be a real analytic group, and $\mathcal{F}^\infty(G)$ the space of real valued differentiable functions on G . The space $\mathcal{F}^\infty(G)$, equipped with the topology of uniform convergence on compact subsets of the functions and of their derivatives, is Fréchet [34, Sect. I.10], and nuclear [34, Sect. III.51]. As a result, $\mathcal{F}^\infty(G)$ is a well-behaved Hopf algebra. See also [2, Ex. 2.2.1], and [1, Ex. 1.7].

We next record below the definitions of topological modules and co-modules, see [33].

Definition 2.6. Given a topological algebra A , a right topological A -module is a t.v.s. M which is a (unital) right A -module structure $M \times A \longrightarrow M$ that extends to a continuous linear map $M \hat{\otimes}_\pi A \longrightarrow M$. Similarly, given a topological coalgebra C , a left topological C -comodule is a t.v.s. V which is a (counital) left C -comodule $V \longrightarrow C \hat{\otimes}_\pi V$.

An immediate example, that we note here for the later use, is the symmetric algebra $S(\mathfrak{g}^*)$ being a topological $U(\mathfrak{g})$ -module via the coadjoint action.

Let us next recall the notion of a differentiable G -module from [11].

Definition 2.7. A topological G -module V is called a differentiable G -module if the map

$$\rho_v : G \longrightarrow V, \quad \rho_v(g) := v \cdot g$$

is differentiable for any $v \in V$.

We then have the following characterization.

Proposition 2.8. Let G be a real analytic group, and V a t.v.s. If V is a left $\mathcal{F}^\infty(G)$ -comodule, via $v \mapsto v^{<-1>} \widehat{\otimes}_\pi v^{<0>} \in \mathcal{F}^\infty(G) \widehat{\otimes}_\pi V$, then V is a differentiable right G -module by

$$v \cdot g := v^{<-1>}(g)v^{<0>}, \quad \forall v \in V, g \in G. \quad (2.1)$$

Conversely, if V is a differentiable right G -module, then V is a left $\mathcal{F}^\infty(G)$ -comodule by

$$\nabla : V \longrightarrow \mathcal{F}^\infty(G) \widehat{\otimes}_\pi V, \quad \nabla(v)(g) := v \cdot g. \quad (2.2)$$

Proof. Let V be a left $\mathcal{F}^\infty(G)$ -comodule. We first show that (2.1) indeed defines an action. To this end, it is enough to observe that

$$\begin{aligned} v \cdot (gg') &= v^{<-1>}(gg')v^{<0>} = \Delta(v^{<-1>})(g, g') \widehat{\otimes}_\pi v^{<0>} = \\ &= (v^{<-1>} \widehat{\otimes}_\pi v^{<0>}^{<-1>})(g, g')v^{<0>}^{<0>} = \\ &= v^{<-1>}(g)v^{<0>}^{<-1>}(g')v^{<0>}^{<0>} = (v \cdot g) \cdot g'. \end{aligned} \quad (2.3)$$

Let us now show that the action (2.1) is a differentiable G -action. For any fixed $v \in V$, we show that the map $\rho_v : G \longrightarrow V$, given by $\rho_v(g) = v \cdot g$, is differentiable. For any $X \in \mathfrak{g}$ we have

$$\begin{aligned} \rho_v(g \exp(tX)) &= v^{<-1>}(g \exp(tX))v^{<0>} = \\ &= v^{<-1>}(g)v^{<0>} + t \widetilde{v^{<-1>}}(g, X, t)v^{<0>} = \rho_v(g) + t\tilde{\rho}_v(g, X, t), \end{aligned}$$

where

$$\tilde{\rho}_v(g, X, t) := \widetilde{v^{<-1>}}(g, X, t)v^{<0>}.$$

This observation ensures the first condition of Definition 2.4. Next we observe for any $Y \in \mathfrak{g}$ that

$$\begin{aligned} Y(\rho_v)(g \exp(tX)) &= \tilde{\rho}_v(g \exp(tX), Y, 0) = \\ &= \widetilde{v^{<-1>}}(g \exp(tX), Y, 0)v^{<0>} = Y(v^{<-1>})(g \exp(tX))v^{<0>} = \\ &= Y(v^{<-1>})(g)v^{<0>} + tY(\widetilde{v^{<-1>}})(g, X, t)v^{<0>} = \\ &= Y(\rho_v)(g) + tY(\tilde{\rho}_v)(g, X, t)v^{<0>}. \end{aligned}$$

Hence, the second requirement of Definition 2.4 also holds for the map $\rho_v : G \longrightarrow V$. We conclude that the G -action is differentiable. Conversely, let V be a differentiable right G -module. It follows at once from the argument (2.3) that (2.2) defines a left $\mathcal{F}^\infty(G)$ -coaction, provided $\nabla(v) : G \longrightarrow V$ is differentiable. The claim, then, follows at once from $\nabla(v) = \rho_v$, for any $v \in V$. \square

2.2 Topological Lie-Hopf algebras

In this subsection we revisit the theory of Lie-Hopf algebras [30, 29] within the framework of topological Hopf algebras. We begin with the matched pair of (topological) Hopf algebras [17].

Let \mathcal{U} and \mathcal{F} be two (topological) Hopf algebras. A right coaction

$$\blacktriangledown : \mathcal{U} \longrightarrow \mathcal{U} \hat{\otimes}_\pi \mathcal{F}, \quad \blacktriangledown(u) = u_{<0>} \hat{\otimes}_\pi u_{<1>}$$

equips \mathcal{U} with a right \mathcal{F} -comodule coalgebra structure if the conditions

$$\begin{aligned} u_{<0> (1)} \hat{\otimes}_\pi u_{<0> (2)} \hat{\otimes}_\pi u_{<1>} &= u_{(1) <0>} \hat{\otimes}_\pi u_{(2) <0>} \hat{\otimes}_\pi u_{(1) <1>} u_{(2) <1>}, \\ \varepsilon(u_{<0>}) u_{<1>} &= \varepsilon(u) 1, \end{aligned}$$

are satisfied for any $u \in \mathcal{U}$. One then forms a cocrossed product topological coalgebra $\mathcal{F} \blacktriangleleft_\pi \mathcal{U}$, that has $\mathcal{F} \hat{\otimes}_\pi \mathcal{U}$ as the underlying t.v.s. and

$$\begin{aligned} \Delta(f \blacktriangleleft_\pi u) &= f_{(1)} \blacktriangleleft_\pi u_{(1) <0>} \hat{\otimes}_\pi f_{(2)} u_{(1) <1>} \blacktriangleleft_\pi u_{(2)}, \\ \varepsilon(f \blacktriangleleft_\pi u) &= \varepsilon(f) \varepsilon(u), \end{aligned}$$

as the topological coalgebra structure.

On the other hand, \mathcal{F} is called a left \mathcal{U} -module algebra if \mathcal{U} acts on \mathcal{F}

$$\triangleright : \mathcal{U} \hat{\otimes}_\pi \mathcal{F} \longrightarrow \mathcal{F}$$

such that

$$u \triangleright 1 = \varepsilon(u) 1, \quad u \triangleright (fg) = (u_{(1)} \triangleright f)(u_{(2)} \triangleright g)$$

for any $u \in \mathcal{U}$, and any $f, g \in \mathcal{F}$. This time one endows the t.v.s. $\mathcal{F} \hat{\otimes}_\pi \mathcal{U}$ with an algebra structure, which is denoted by $\mathcal{F} \hat{\bowtie}_\pi \mathcal{U}$, with $1 \hat{\bowtie}_\pi 1$ as its unit and the multiplication given by

$$(f \hat{\bowtie}_\pi u)(g \hat{\bowtie}_\pi v) = f(u_{(1)} \triangleright g) \hat{\bowtie}_\pi u_{(2)} v.$$

Finally, the pair $(\mathcal{F}, \mathcal{U})$ of topological Hopf algebras is called a matched pair of Hopf algebras if \mathcal{U} is a right \mathcal{F} -comodule coalgebra, \mathcal{F} is a left \mathcal{U} -module algebra, and

$$\begin{aligned}\varepsilon(u \triangleright f) &= \varepsilon(u)\varepsilon(f), \\ \Delta(u \triangleright f) &= u_{(1) < 0 >} \triangleright f_{(1)} \hat{\otimes}_\pi u_{(1) < 1 >} (u_{(2)} \triangleright f_{(2)}), \\ \nabla(1) &= 1 \otimes 1, \\ \nabla(uv) &= u_{(1) < 0 >} v_{< 0 >} \hat{\otimes}_\pi u_{(1) < 1 >} (u_{(2)} \triangleright v_{< 1 >}), \\ u_{(2) < 0 >} \hat{\otimes}_\pi (u_{(1)} \triangleright f) u_{(2) < 1 >} &= u_{(1) < 0 >} \hat{\otimes}_\pi u_{(1) < 1 >} (u_{(2)} \triangleright f),\end{aligned}$$

for any $u \in \mathcal{U}$, and any $f \in \mathcal{F}$. One then forms the bicrossed product Hopf algebra $\mathcal{F} \blacktriangleright_{\hat{\otimes}_\pi} \mathcal{U}$. It has $\mathcal{F} \blacktriangleleft_\pi \mathcal{U}$ as the underlying coalgebra, $\mathcal{F} \hat{\times}_\pi \mathcal{U}$ as the underlying algebra, and its antipode is defined by

$$S(f \hat{\times}_\pi u) = (1 \hat{\triangleleft}_\pi S(u_{< 0 >}))(S(fu_{< 1 >})) \hat{\triangleleft}_\pi 1, \quad \forall f \in \mathcal{F}, \forall u \in \mathcal{U}.$$

Definition 2.9. A left topological \mathfrak{g} -module over a topological Lie algebra \mathfrak{g} is a t.v.s. V such that the left \mathfrak{g} -module structure map $\mathfrak{g} \hat{\otimes}_\pi V \longrightarrow V$ is continuous.

Example 2.10. Let \mathfrak{g} be a topological Lie algebra. The symmetric algebra $S(\mathfrak{g}^*)$ is a (right) topological \mathfrak{g} -module by the coadjoint action via [1, Coroll. A.2.8].

Now let \mathcal{F} be a commutative topological Hopf algebra on which a topological Lie algebra \mathfrak{g} acts by derivations. We endow the vector space $\mathfrak{g} \hat{\otimes}_\pi \mathcal{F}$ with the following bracket:

$$[X \hat{\otimes}_\pi f, Y \hat{\otimes}_\pi g] = [X, Y] \hat{\otimes}_\pi fg + Y \hat{\otimes}_\pi \varepsilon(f)X \triangleright g - X \hat{\otimes}_\pi \varepsilon(g)Y \triangleright f. \quad (2.4)$$

Lemma 2.11. Let a topological Lie algebra \mathfrak{g} act on a commutative topological Hopf algebra \mathcal{F} by derivations, and $\varepsilon(X \triangleright f) = 0$ for any $X \in \mathfrak{g}$ and $f \in \mathcal{F}$. Then the bracket (2.4) endows $\mathfrak{g} \hat{\otimes}_\pi \mathcal{F}$ with a topological Lie algebra structure.

Proof. It is checked in [30], see also [29], that the bracket is antisymmetric, and that the Jacobi identity is satisfied. Moreover, since the bracket on \mathfrak{g} , and the action of \mathfrak{g} on \mathcal{F} are continuous, it follows that the bracket (2.4) is also continuous. \square

Next, let \mathcal{F} coacts on \mathfrak{g} via $\nabla_{\mathfrak{g}} : \mathfrak{g} \rightarrow \mathfrak{g} \hat{\otimes}_{\pi} \mathcal{F}$. Using the action of \mathfrak{g} on \mathcal{F} , and the coaction of \mathcal{F} on \mathfrak{g} , we define an action of \mathfrak{g} on $\mathcal{F} \hat{\otimes}_{\pi} \mathcal{F}$ by

$$X \bullet (f^1 \hat{\otimes}_{\pi} f^2) = X_{<0>} \triangleright f^1 \hat{\otimes}_{\pi} X_{<1>} f^2 + f^1 \hat{\otimes}_{\pi} X \triangleright f^2. \quad (2.5)$$

We note that since the \mathcal{F} -coaction on \mathfrak{g} , and the \mathfrak{g} action on \mathcal{F} are continuous, the action (2.5) is also continuous.

Definition 2.12. *Let a topological Lie algebra \mathfrak{g} act on a commutative topological Hopf algebra \mathcal{F} by continuous derivations, and \mathcal{F} coacts on \mathfrak{g} continuously. We say that \mathcal{F} is a topological \mathfrak{g} -Hopf algebra if*

1. *the coaction $\nabla_{\mathfrak{g}} : \mathfrak{g} \rightarrow \mathfrak{g} \hat{\otimes}_{\pi} \mathcal{F}$ is a map of topological Lie algebras,*
2. *Δ and ε are \mathfrak{g} -linear, i.e., $\Delta(X \triangleright f) = X \bullet \Delta(f)$, $\varepsilon(X \triangleright f) = 0$, for any $f \in \mathcal{F}$, and any $X \in \mathfrak{g}$.*

Following [30] we extend the \mathcal{F} -coaction from \mathfrak{g} to $U(\mathfrak{g})$.

Lemma 2.13. *The extension of the coaction $\nabla_{\mathfrak{g}} : \mathfrak{g} \rightarrow \mathfrak{g} \hat{\otimes}_{\pi} \mathcal{F}$ to $\nabla : U(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \hat{\otimes}_{\pi} \mathcal{F}$, via*

$$\begin{aligned} \nabla(uu') &= u_{(1) <0>} u'_{<0>} \hat{\otimes}_{\pi} u_{(1) <1>} (u_{(2)} \triangleright u'_{<1>}), \quad \forall u, u' \in U(\mathfrak{g}), \\ \nabla(1) &= 1 \hat{\otimes}_{\pi} 1, \end{aligned} \quad (2.6)$$

is well-defined.

Proof. It is checked in [30] that (2.6) is well-defined. Hence, we just need to show that it is continuous, which follows from the linearity [1, Lemma A.2.2]. \square

As a result, we obtain a topological version of [30, Thm. 2.6] as follows.

Theorem 2.14. *Let \mathcal{F} be a commutative topological \mathfrak{g} -Hopf algebra. Then via the coaction of \mathcal{F} on $U(\mathfrak{g})$ defined above and the natural action of $U(\mathfrak{g})$ on \mathcal{F} , the pair $(U(\mathfrak{g}), \mathcal{F})$ becomes a matched pair of topological Hopf algebras. Conversely, for a commutative topological Hopf algebra \mathcal{F} , if $(U(\mathfrak{g}), \mathcal{F})$ is a matched pair of topological Hopf algebras then \mathcal{F} is a topological \mathfrak{g} -Hopf algebra.*

Example 2.15. Let (G_1, G_2) be a matched pair of real analytic groups with mutual analytical actions, and let $(\mathfrak{g}_1, \mathfrak{g}_2)$ be their Lie algebras. Since the infinitesimal action

$$\varphi_X : G_2 \longrightarrow \mathfrak{g}_1, \quad \varphi_X(y) := \left. \frac{d}{ds} \right|_{s=0} y \triangleright \exp(sX)$$

of G_2 on \mathfrak{g}_1 is differentiable¹, it follows from Proposition 2.8 that \mathfrak{g}_1 is a right $\mathcal{F}^\infty(G_2)$ -comodule. Moreover, \mathfrak{g}_1 acts on $\mathcal{F}^\infty(G_2)$ by

$$X(f) := \left. \frac{d}{ds} \right|_{s=0} \exp(sX) \triangleright f,$$

for any $X \in \mathfrak{g}_1$, and any $f \in \mathcal{F}^\infty(G_2)$. Indeed,

$$\begin{aligned} X(f)(y \exp(tY)) &= \left. \frac{d}{ds} \right|_{s=0} (\exp(sX) \triangleright f)((y \exp(tY))) = \\ &= \left. \frac{d}{ds} \right|_{s=0} f((y \exp(tY)) \triangleleft \exp(sX)) = \\ &= \left. \frac{d}{ds} \right|_{s=0} f((y \triangleleft (\exp(tY) \triangleright \exp(sX))) (\exp(tY) \triangleleft \exp(sX))) = \\ &= \left. \frac{d}{ds} \right|_{s=0} f((y \triangleleft (\exp(tY) \triangleright \exp(sX))) \exp(tY)) + \\ &\quad \left. \frac{d}{ds} \right|_{s=0} f(y(\exp(tY) \triangleleft \exp(sX))) \end{aligned}$$

for any $y \in G_2$, and $Y \in \mathfrak{g}_2$, and $f(y \exp(tY)) = f(y) + t f'(y, Y, t)$ for some continuous $f' : G_2 \times \mathfrak{g}_2 \times \mathbb{R} \rightarrow \mathbb{R}$ with $f'(y, Y, 0) = Y(f')(y)$, [11]. Therefore,

$$\begin{aligned} X(f)(y \exp(tY)) &= \left. \frac{d}{ds} \right|_{s=0} f(y \triangleleft (\exp(tY) \triangleright \exp(sX))) + t \varphi(y, Y, t) = \\ &= (\exp(tY) \triangleright X)(f)(y) + t \varphi(y, Y, t) = X(f)(y) + t \psi(X, Y, t)(f) + t \varphi(y, Y, t), \end{aligned}$$

where $\psi(X, Y, 0) = Y \triangleright X$, and

$$\begin{aligned} \varphi(y, Y, t) &:= \\ &= \left. \frac{d}{ds} \right|_{s=0} f'((y \triangleleft (\exp(tY) \triangleright \exp(sX))), Y, t) + \left. \frac{d}{ds} \right|_{s=0} f''(y, Y \triangleleft \exp(sX), t) \end{aligned}$$

with $f'' : G_2 \times \mathfrak{g}_2 \times \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$f''(y, Y \triangleleft \exp(sX), 0) = (Y \triangleleft \exp(sX))(f)(y).$$

¹Follows from

$$\exp(tY) \triangleright X = X + t \mu(X, Y, t), \quad \mu(X, Y, 0) = Y \triangleright X,$$

for any $X \in \mathfrak{g}_1$, and any $Y \in \mathfrak{g}_2$, which, in turn, follows from the action of G_2 on G_1 being analytical.

As a result, $X(f)(y \exp(tY)) = X(f)(y) + t\Phi(y, Y, t)$, where

$$\Phi(y, Y, 0) = (Y \triangleright X)(f)(y) + \frac{d}{ds} \Big|_{s=0} f'(y \triangleleft \exp(sX), Y, 0) + (Y \triangleleft X)(f)(y),$$

which is differentiable. We thus obtain that $X(f) \in \mathcal{F}^\infty(G_2)$ for any $f \in \mathcal{F}^\infty(G_2)$, and $X \in \mathfrak{g}_1$. Next, it remains to verify the conditions of Definition 2.12. The first requirement follows from [16, Lemma 3.2], and the second from [16, Thm. 3.1]. It then follows that $\mathcal{F}^\infty(G_2)$ is a \mathfrak{g}_1 -Hopf algebra.

Finally, in view of Theorem 2.14 we obtain the Hopf algebra $\mathcal{H} := \mathcal{F}^\infty(G_2) \blacktriangleright_\pi U(\mathfrak{g}_1)$.

3 Hopf-cyclic cohomology for topological Hopf algebras

In this section we first recall the basic definitions and results for Hopf-cyclic cohomology with coefficients in the category of topological Hopf algebras. We continue by characterizing the category of coefficient spaces (SAYD modules) over a topological Hopf algebra \mathcal{H} as the category of representations of a topological algebra associated to the Hopf algebra \mathcal{H} .

3.1 Hopf-cyclic complex for topological Hopf algebras

We shall include, in this subsection, a brief discussion of Hopf-cyclic cohomology with coefficients in the category of topological Hopf algebras. To this end, we adopt the categorical viewpoint of [4], see also [15], to consider cocyclic modules in the category of topological Hopf algebras.

Let \mathcal{H} be a topological Hopf algebra. A character $\delta : \mathcal{H} \rightarrow k$ is a continuous unital algebra map, and a group-like element $\sigma \in \mathcal{H}$ is the dual object of the character, *i.e.* $\Delta(\sigma) = \sigma \widehat{\otimes}_\pi \sigma$ and $\varepsilon(\sigma) = 1$. The pair (δ, σ) is called a modular pair in involution (MPI) if

$$\delta(\sigma) = 1, \quad \text{and} \quad S_\delta^2 = \text{Ad}_\sigma,$$

where $\text{Ad}_\sigma(h) = \sigma h \sigma^{-1}$, and $S_\delta(h) = \delta(h_{(1)})S(h_{(2)})$.

In the presence of a topology, we shall extend the scope of the canonical MPI associated to a Lie-Hopf algebra [29, Thm. 3.2], to a \mathfrak{g} -Hopf algebra of an infinite dimensional Lie algebra \mathfrak{g} .

It follows from [2, Prop. 1(iii)] that $\mathfrak{g}^\circ \widehat{\otimes}_\pi \mathfrak{g} \cong \text{End}(\mathfrak{g})$, and hence there exists a well-defined element

$$\rho = \sum_{i \in I} f^i \widehat{\otimes}_\pi X_i \in \mathfrak{g}^\circ \widehat{\otimes}_\pi \mathfrak{g},$$

that corresponds to $\text{Id} \in \text{End}(\mathfrak{g})$. This, in turn, results in a well-defined functional

$$\delta_{\mathfrak{g}} := \sum_{i \in I} X_i \triangleright f^i : \mathfrak{g} \longrightarrow k. \quad (3.1)$$

Note that in case \mathfrak{g} is finite dimensional, $\{X_i \mid i \in I\}$ and $\{f^i \mid i \in I\}$ are dual pair of bases, and we have $\delta_{\mathfrak{g}} = \text{Tr} \circ \text{ad}$.

Next, let $\mathfrak{g} = \cup_{n \in \mathbb{N}} V_n$ be a sequence of definition [34, Sect. 13], compatible with the \mathcal{F} -coaction, that is, the right \mathcal{F} -coaction $\nabla : \mathfrak{g} \rightarrow \mathfrak{g} \widehat{\otimes}_\pi \mathcal{F}$ restricts to $\nabla : V_n \rightarrow V_n \widehat{\otimes}_\pi \mathcal{F}$, for any $n \in \mathbb{N}$.

For instance, if (G_1, G_2) be a matched pair of Lie groups, with Lie algebras \mathfrak{g}_1 and \mathfrak{g}_2 , such that \mathfrak{g}_1 is a locally finite (differentiable) G_2 -module¹, then $\mathfrak{g}_1 = \sum_{n \in \mathbb{N}} V_n$ for the finite dimensional G_2 -submodules V_n , $n \in \mathbb{N}$, and $\nabla : \mathfrak{g}_1 \rightarrow \mathfrak{g}_1 \widehat{\otimes}_\pi \mathcal{F}^\infty(G_2)$ restricts to $\nabla : V_n \rightarrow V_n \widehat{\otimes}_\pi \mathcal{F}^\infty(G_2)$ for any $n \in \mathbb{N}$.

For any such topological \mathfrak{g} -Hopf algebra \mathcal{F} , let

$$\sigma = \lim_{n \rightarrow \infty} \sigma_n \in \mathcal{F}, \quad (3.2)$$

where $\sigma_n \in \mathcal{F}$ is defined as the determinant of the first order matrix coefficients on the finite dimensional $\cup_{k=1}^n V_k \subseteq \mathfrak{g}$. By [29, Lemma 3.1], σ_n is a group-like for any $n \in \mathbb{N}$, and since the comultiplication $\Delta : \mathcal{F} \rightarrow \mathcal{F} \widehat{\otimes}_\pi \mathcal{F}$ is continuous, we have $\Delta(\sigma) = \Delta(\lim_{n \rightarrow \infty} \sigma_n) = \lim_{n \rightarrow \infty} \Delta(\sigma_n) = \sigma \widehat{\otimes}_\pi \sigma$. That is, $\sigma \in \mathcal{F}$ is also a group-like.

Lemma 3.1. *The group-like $\sigma \in \mathcal{F}$ is independent of the choice of the sequence of definition.*

Proof. Let $\mathfrak{g} = \cup_{n \in \mathbb{N}} V_n = \cup_{n \in \mathbb{N}} W_n$ be two sequence of definitions. Accordingly we have two sequences $(\sigma_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}$, obtained as the

¹Any element of \mathfrak{g}_1 is contained in a finite dimensional (differentiable) G_2 -module [12, Sect. 1.2].

first order matrix coefficients of the coaction $V_n \longrightarrow V_n \widehat{\otimes}_\pi \mathcal{F}$, and $(\sigma'_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}$, obtained as the first order matrix coefficients of the coaction $W_n \longrightarrow W_n \widehat{\otimes}_\pi \mathcal{F}$. Let then $\sigma := \lim_{n \rightarrow \infty} \sigma_n \in \mathcal{F}$, and similarly $\sigma' = \lim_{n \rightarrow \infty} \sigma'_n \in \mathcal{F}$.

Since a sequence of definition consists of an increasing sequence of subspaces, for any $n \in \mathbb{N}$, there is $n_0 \in \mathbb{N}$ such that $V_n \subseteq W_{n_0}$, and hence $\sigma' \in \mathcal{F}$ agrees with $\sigma \in \mathcal{F}$ on V_n for any $n \in \mathbb{N}$. We thus conclude that $\sigma = \sigma'$. \square

Accordingly, we have the following generalization of [29, Thm. 3.2].

Theorem 3.2. *Let \mathfrak{g} be a topological Lie algebra, and \mathcal{F} a topological \mathfrak{g} -Hopf algebra. Let also \mathfrak{g} have a sequence of definition compatible with the \mathcal{F} -coaction. Then $\delta : U(\mathfrak{g}) \longrightarrow k$ being the extension of the functional $\delta_{\mathfrak{g}} : \mathfrak{g} \longrightarrow k$ of (3.1), and $\sigma \in \mathcal{F}$ the group-like defined by (3.2), the pair (δ, σ) is an MPI for the Hopf algebra $\mathcal{F} \blacktriangleright_{\pi} U(\mathfrak{g})$.*

Proof. Since a sequence of definition consists of an increasing sequence of subspaces, for any $X_i \in \mathfrak{g}$ there is $n_0 \in \mathbb{N}$ such that $X_i \in \cup_{n=1}^{n_0} V_i$, which is finite dimensional. The claim then follows from the proof of [29, Thm. 3.2], see also [30, Thm. 3.2]. \square

Stable anti-Yetter-Drinfeld (SAYD) modules appeared first in [10, 13] as the generalizations of modular pairs in involution [5]. In the rest of this subsection we upgrade them to the level of topological Hopf algebras.

Definition 3.3. *Let V be a topological right \mathcal{H} -module by $V \widehat{\otimes} \mathcal{H} \rightarrow V$, $v \widehat{\otimes} h \rightarrow v \cdot h$, and left \mathcal{H} -comodule via $\blacktriangledown : V \rightarrow \mathcal{H} \widehat{\otimes} V$, $\blacktriangledown(v) = v_{<-1>} \widehat{\otimes} v_{<0>}$. We say that V is an AYD module over \mathcal{H} if*

$$\blacktriangledown(v \cdot h) = S(h_{(3)})v_{<-1>}h_{(1)} \widehat{\otimes}_\pi v_{<0>} \cdot h_{(2)},$$

for any $v \in V$ and $h \in \mathcal{H}$. Moreover, V is called stable if

$$v_{<0>} \cdot v_{<-1>} = v,$$

for any $v \in V$.

Similar to the algebraic case, any MPI defines a one dimensional SAYD module and all one dimensional SAYD modules come this way.

Proposition 3.4. *Let \mathcal{H} be a topological Hopf algebra, $\sigma \in \mathcal{H}$ a group-like element, and $\delta \in \mathcal{H}^\circ$ a character. Then, (δ, σ) is an MPI if and only if ${}^\sigma k_\delta$ is a SAYD module over \mathcal{H} .*

We next define the Hopf-cyclic cohomology, with SAYD coefficients, of topological module coalgebras. To this end we first recall the tensor product of topological modules over a topological algebra from [33, Def. 1.7]. Let A be a topological algebra, M a topological right A -module, and N a topological left A -module. Then, $M \widehat{\otimes}_A N$ is defined to be the vector space $(M \times N)/W$, equipped with the quotient topology, where

$$W := \text{Span}\{(ma, n) - (m, an) \mid m \in M, n \in N, a \in A\} \subseteq M \times N.$$

We note also from [34, Prop. 4.5] that, as a topological space, $M \widehat{\otimes}_A N$ is Hausdorff if and only if W is closed, see also [33, Prop. 1.5].

Let V be a right-left SAYD module over a topological Hopf algebra \mathcal{H} , and \mathcal{C} a topological \mathcal{H} -module coalgebra via the continuous action $\mathcal{H} \widehat{\otimes}_\pi \mathcal{C} \longrightarrow \mathcal{C}$, that is,

$$\Delta(h \cdot c) = (h_{(1)} \cdot c_{(1)}) \widehat{\otimes}_\pi (h_{(2)} \cdot c_{(2)}), \quad \varepsilon(h \cdot c) = \varepsilon(h)\varepsilon(c).$$

We then define the topological Hopf-cyclic complex of \mathcal{C} , with coefficients in V , under the symmetry of \mathcal{H} by

$$C_{\text{top}}^n(\mathcal{C}, \mathcal{H}, V) := V \widehat{\otimes}_\mathcal{H} \mathcal{C}^{\widehat{\otimes}_\pi n+1}, \quad n \geq 0,$$

equipped with the face operators

$$\begin{aligned} \partial_i : C_{\text{top}}^n(\mathcal{C}, \mathcal{H}, V) &\longrightarrow C_{\text{top}}^{n+1}(\mathcal{C}, \mathcal{H}, V), \quad 0 \leq i \leq n+1, \\ \partial_i(v \widehat{\otimes}_\mathcal{H} c^0 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi c^n) &= v \widehat{\otimes}_\mathcal{H} c^0 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi c^i_{(1)} \widehat{\otimes}_\pi c^i_{(2)} \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi c^n, \\ \partial_{n+1}(v \widehat{\otimes}_\mathcal{H} c^0 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi c^n) &= v_{<0>} \widehat{\otimes}_\pi c^0_{(2)} \widehat{\otimes}_\pi c^1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi c^n \widehat{\otimes}_\pi v_{<-1>} \cdot c^0_{(1)}, \end{aligned}$$

the degeneracy operators

$$\begin{aligned} \sigma_j : C_{\text{top}}^n(\mathcal{C}, \mathcal{H}, V) &\longrightarrow C_{\text{top}}^{n-1}(\mathcal{C}, \mathcal{H}, V), \quad 0 \leq j \leq n-1, \\ \sigma_j(v \widehat{\otimes}_\mathcal{H} c^0 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi c^n) &= v \widehat{\otimes}_\mathcal{H} c^0 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi \varepsilon(c^{j+1}) \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi c^n, \end{aligned}$$

and the cyclic operator

$$\begin{aligned} \tau : C_{\text{top}}^n(\mathcal{C}, \mathcal{H}, V) &\longrightarrow C_{\text{top}}^n(\mathcal{C}, \mathcal{H}, V), \\ \tau(v \widehat{\otimes}_\mathcal{H} c^0 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi c^n) &= v_{<0>} \widehat{\otimes}_\mathcal{H} c^1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi v_{<-1>} \cdot c^0. \end{aligned} \quad (3.3)$$

Then, the graded module $C_{\text{top}}^*(\mathcal{C}, \mathcal{H}, V)$ becomes a cocyclic module [9], with the Hochschild coboundary

$$b : C_{\text{top}}^n(\mathcal{C}, \mathcal{H}, V) \longrightarrow C_{\text{top}}^{n+1}(\mathcal{C}, \mathcal{H}, V), \quad b := \sum_{i=0}^{n+1} (-1)^i \partial_i,$$

and the Connes boundary operator

$$B : C_{\text{top}}^n(\mathcal{C}, \mathcal{H}, V) \longrightarrow C_{\text{top}}^{n-1}(\mathcal{C}, \mathcal{H}, V), \quad B = \sum_{k=0}^n (-1)^{kn} \tau^k \sigma_{-1},$$

which satisfy $b^2 = B^2 = (b + B)^2 = 0$ by [4]. We note that

$$\sigma_{-1} = \sigma_{n-1} \circ \tau : C_{\text{top}}^n(\mathcal{C}, \mathcal{H}, V) \longrightarrow C_{\text{top}}^{n-1}(\mathcal{C}, \mathcal{H}, V)$$

is the extra degeneracy operator.

Definition 3.5. *The Hopf-cyclic cohomology of the topological coalgebra \mathcal{C} under the \mathcal{H} -module coalgebra symmetry, with coefficients in a SAYD module V over \mathcal{H} , is denoted by $HC_{\text{top}}^*(\mathcal{C}, \mathcal{H}, V)$, and is defined to be the total cohomology of the bicomplex*

$$CC_{\text{top}}^{p,q}(\mathcal{C}, \mathcal{H}, V) = \begin{cases} V \widehat{\otimes}_{\mathcal{H}} \mathcal{C}^{\widehat{\otimes}_{\pi} q-p}, & \text{if } 0 \leq p \leq q, \\ 0, & \text{otherwise.} \end{cases}$$

The periodic Hopf-cyclic cohomology of the topological coalgebra \mathcal{C} under the \mathcal{H} -module coalgebra symmetry, with coefficients in a SAYD module V over \mathcal{H} , is denoted by $HP_{\text{top}}^(\mathcal{C}, \mathcal{H}, V)$, and it is defined to be the total cohomology of the bicomplex*

$$CC_{\text{top}}^{p,q}(\mathcal{C}, \mathcal{H}, V) = \begin{cases} V \widehat{\otimes}_{\mathcal{H}} \mathcal{C}^{\widehat{\otimes}_{\pi} q-p}, & \text{if } p \leq q, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, it follows from the proof of [33, Prop. 1.5] that, in case $\mathcal{C} = \mathcal{H}$ a module coalgebra with the left regular action of \mathcal{H} , the map

$$\begin{aligned} \mathcal{I} : V \widehat{\otimes}_{\mathcal{H}} \mathcal{H}^{\widehat{\otimes} n+1} &\longrightarrow V \widehat{\otimes}_{\pi} \mathcal{H}^{\widehat{\otimes}_{\pi} n} \\ v \widehat{\otimes}_{\mathcal{H}} h^0 \widehat{\otimes}_{\pi} \cdots \widehat{\otimes}_{\pi} h^n &\mapsto v \cdot h^0_{(1)} \widehat{\otimes}_{\pi} S(h^0_{(2)}) (h^1 \widehat{\otimes}_{\pi} \cdots \widehat{\otimes}_{\pi} h^n) \end{aligned} \quad (3.4)$$

identifies $C_{\text{top}}^n(\mathcal{C}, \mathcal{H}, V) = V \widehat{\otimes}_{\mathcal{H}} \mathcal{H}^{\widehat{\otimes} n+1}$ and $C_{\text{top}}^n(\mathcal{H}, V) := V \widehat{\otimes}_{\pi} \mathcal{H}^{\widehat{\otimes}_{\pi} n}$ as topological spaces. In this case, the cocyclic structure is transformed into the one with the face operators

$$\begin{aligned} \partial_i : C_{\text{top}}^n(\mathcal{H}, V) &\rightarrow C_{\text{top}}^{n+1}(\mathcal{H}, V), \quad 0 \leq i \leq n+1, \\ \partial_0(v \widehat{\otimes}_{\pi} h^1 \widehat{\otimes}_{\pi} \cdots \widehat{\otimes}_{\pi} h^n) &= v \otimes 1 \otimes h^1 \widehat{\otimes}_{\pi} \cdots \widehat{\otimes}_{\pi} h^n, \\ \partial_i(v \otimes h^1 \widehat{\otimes}_{\pi} \cdots \widehat{\otimes}_{\pi} h^n) &= v \otimes h^1 \widehat{\otimes}_{\pi} \cdots \widehat{\otimes}_{\pi} h^i h^i_{(1)} \otimes h^i_{(2)} \widehat{\otimes}_{\pi} \cdots \widehat{\otimes}_{\pi} h^n, \\ \partial_{n+1}(v \otimes h^1 \widehat{\otimes}_{\pi} \cdots \widehat{\otimes}_{\pi} h^n) &= v_{<0>} \otimes h^1 \widehat{\otimes}_{\pi} \cdots \widehat{\otimes}_{\pi} h^n \otimes v_{<-1>}, \end{aligned}$$

the degeneracy operators

$$\begin{aligned}\sigma_j : C_{\text{top}}^n(\mathcal{H}, V) &\rightarrow C_{\text{top}}^{n-1}(\mathcal{H}, V), \quad 0 \leq j \leq n-1, \\ \sigma_j(v \otimes h^1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi h^n) &= v \otimes h^1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi \varepsilon(h^{j+1}) \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi h^n,\end{aligned}$$

and the cyclic operator

$$\begin{aligned}\tau : C_{\text{top}}^n(\mathcal{H}, V) &\rightarrow C_{\text{top}}^n(\mathcal{H}, V), \\ \tau(v \otimes h^1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi h^n) &= v_{<0>} h^1_{(1)} \otimes S(h^1_{(2)}) \cdot (h^2 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi h^n \otimes v_{<-1>}).\end{aligned}$$

3.2 Characterization of Hopf-cyclic coefficients

In this subsection we upgrade the content of [10, Prop. 4.2] to topological Hopf algebras. More precisely, we shall identify the category of SAYD modules, over a topological Hopf algebra \mathcal{H} , with the representation category of an algebra $B_{AYD}(\mathcal{H})$ associated to \mathcal{H} .

To this end, we recall from [10, Prop. 4.2] that in case H is finite dimensional, V is a AYD module over H if and only if it is a module over $B_{AYD}(H) := H^* \otimes H$, the algebra structure of which is given by

$$(\varphi \otimes h) \cdot (\varphi' \otimes h') = \varphi'_{(1)}(S^{-1}(h_{(3)}))\varphi'_{(3)}(S^2(h_{(1)}))\varphi\varphi'_{(2)} \otimes h_{(2)}h'. \quad (3.5)$$

Now let \mathcal{H} be a topological Hopf algebra, \mathcal{H}° is its (strong) dual, and $B_{AYD}^{\text{top}}(\mathcal{H}) := \mathcal{H}^\circ \widehat{\otimes}_\pi \mathcal{H}$ is the algebra whose multiplication is the extension of (3.5) to $\widehat{\otimes}_\pi$.

It follows from [2, Prop. 1(iii)], see also [1], that there is an isomorphism

$$\lambda : B_{AYD}^{\text{top}}(\mathcal{H}) \longrightarrow \text{End}(\mathcal{H}), \quad \lambda(\varphi \widehat{\otimes}_\pi h)(x) = \varphi(x)h$$

of vector spaces. As a result, we have an element

$$\rho := \sum_{i \in I} f^i \widehat{\otimes}_\pi x_i \in B_{AYD}^{\text{top}}(\mathcal{H}) \quad (3.6)$$

corresponding to the identity $\text{Id}_{\mathcal{H}} \in \text{End}(\mathcal{H})$, i.e. $\lambda(\rho) = \text{Id}_{\mathcal{H}}$. We thus record below the following generalization of [10, Prop. 4.2] as the main result of the present subsection.

Proposition 3.6. *Let \mathcal{H} be a topological Hopf algebra. If V is a right-left AYD module over \mathcal{H} , then it is a right $B_{AYD}^{\text{top}}(\mathcal{H})$ -module via*

$$v \cdot (\varphi \widehat{\otimes}_\pi h) := \varphi(v_{<-1>})v_{<0>} \cdot h.$$

Conversely, if \mathcal{H} is a topological Hopf algebra and V is a right $B_{AYD}^{top}(\mathcal{H})$ -module, then it is a right-left AYD module over \mathcal{H} by the right \mathcal{H} -action

$$v \cdot h := v \cdot (\varepsilon \widehat{\otimes}_\pi h),$$

and the left \mathcal{H} -coaction

$$\nabla_{\mathcal{H}}^L(v) := \sum_{i \in I} x_i \widehat{\otimes}_\pi v \cdot (f^i \widehat{\otimes}_\pi 1).$$

As for the stability we have the following result.

Proposition 3.7. *Let \mathcal{H} be a topological Hopf algebra, and V a right $B_{AYD}^{top}(\mathcal{H})$ -module. Then V is stable if and only if the fixed point set of ρ is V .*

Example 3.8. Let H be a countable dimensional algebraic Hopf algebra, equipped with the natural topology. Then,

1. an AYD module over H ,
2. $H^* \otimes H$,

are right modules over $B_{AYD}(H) = B_{AYD}^{top}(H)$, the former by Proposition 3.6, and the latter by the right regular action. Hence they make coefficient spaces for the topological Hopf-cyclic cohomology, under the symmetry of H .

Remark 3.9. If \mathcal{H} is a topological Hopf algebra, V is a right / left SAYD module over \mathcal{H} , and \mathcal{C} is a topological left \mathcal{H} -module coalgebra, then the cyclic operator (3.3) can be given by the element $\rho \in B_{AYD}^{top}(\mathcal{H})$ of (3.6) as

$$\tau(v \widehat{\otimes}_{\mathcal{H}} c^0 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi c^n) = \sum_{i \in I} (v \cdot f^i) \widehat{\otimes}_{\mathcal{H}} c^1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi x_i \cdot c^0$$

for any $v \in V$, and any $c^0, \dots, c^n \in \mathcal{C}$.

Remark 3.10. If V is an AYD module over a Hopf algebra \mathcal{H} , then its (strong) dual V° is naturally an AYD-contra-module [3], which is, in view of Proposition 3.6, a left module over B_{AYD}^{top} . This leads us to a classification of AYD contra-modules, and has a direct application in cup products in Hopf-cyclic cohomology. This is the ground for a detailed discussion in an upcoming paper [31].

Remark 3.11. We finally note that the topological Hopf-cyclic cohomology is a natural generalization of the algebraic one. Any (countable dimensional) algebraic Hopf algebra H , and any (countable dimensional) algebraic SAYD module V over H can be topologized with the strict inductive limit topology [34] (the natural topology [1, Appendix 2]). In this case, the algebraic and the topological Hopf-cyclic complexes coincide, [1, Prop. A.2.7(ii)].

4 Cyclic cohomology for topological Lie algebras

In this section we review the cyclic cohomology theory for Lie algebras in the presence of a topology. The main difference between the algebraic and the topological points of views manifests itself on the corepresentation categories of a topological Lie algebra \mathfrak{g} , and its universal enveloping algebra $U(\mathfrak{g})$ viewed as a topological Hopf algebra. More precisely, in Subsection 4.1 we remove the local conilpotency condition of [27, Def. 5.4], and we thus cover examples such as the symmetric algebra $S(\mathfrak{g}^*)$, see [27, Ex. 5.6]. In Subsection 4.2, on the other hand, we discuss two cyclic cohomology theories associated to a topological Lie algebra.

4.1 Corepresentations of topological Lie algebras

In this subsection we shall study the categories of comodules over a topological Lie algebra \mathfrak{g} , and comodules over its universal enveloping Hopf algebra $U(\mathfrak{g})$.

Definition 4.1. *Let \mathfrak{g} be a topological Lie algebra, and V a t.v.s. Then V is called a topological left \mathfrak{g} -comodule if there exists a linear map*

$$\nabla_{\mathfrak{g}} : V \longrightarrow \mathfrak{g} \hat{\otimes}_{\pi} V, \quad \nabla_{\mathfrak{g}}(v) := v_{[-1]} \hat{\otimes}_{\pi} v_{[0]}$$

such that

$$v_{[-2]} \wedge v_{[-1]} \hat{\otimes}_{\pi} v_{[0]} = 0.$$

Example 4.2. The symmetric algebra $S(\mathfrak{g}^*)$, equipped with the natu-

ral topology, is a (left) topological \mathfrak{g} -comodule by the Koszul coaction

$$\begin{aligned}\nabla_{\mathfrak{g}}^* : S(\mathfrak{g}^*) &\longrightarrow \mathfrak{g} \widehat{\otimes}_{\pi} S(\mathfrak{g}^*), \\ \nabla_{\mathfrak{g}}^*(R)(Y_0, \dots, Y_q) &= \sum_{k=0}^q Y_k R(Y_0, \dots, \widehat{Y}_k, \dots, Y_q),\end{aligned}\tag{4.1}$$

for any $Y_0, \dots, Y_q \in \mathfrak{g}$, and any $R \in S_q(\mathfrak{g}^*)$, [24].

Let us recall from [27, Prop. 5.2] that any \mathfrak{g} -comodule V is naturally a module over $S(\mathfrak{g}^*)$ via

$$v \cdot f = f(v_{[-1]})v_{[0]}.\tag{4.2}$$

We shall see that this correspondence is revisable in the topological case as well.

Proposition 4.3. *Let \mathfrak{g} be a topological Lie algebra, and V a t.v.s. Then, V is a topological left \mathfrak{g} -comodule if and only if it is a topological right $S(\mathfrak{g}^*)$ -module via (4.2).*

Proof. Let V be a topological left \mathfrak{g} -comodule via $\nabla : V \longrightarrow \mathfrak{g} \widehat{\otimes}_{\pi} V$, $\nabla(v) = v_{[-1]} \widehat{\otimes}_{\pi} v_{[0]}$. Then since the \mathfrak{g} -coaction is continuous by the assumption, the (right) $S(\mathfrak{g}^*)$ -action given by (4.2) is also continuous. Conversely let V be a topological right $S(\mathfrak{g}^*)$ -module via an action $V \widehat{\otimes}_{\pi} S(\mathfrak{g}^*) \longrightarrow V$, $v \widehat{\otimes}_{\pi} f \mapsto v \cdot f$. Using the element $\sum_{i \in I} f^i \widehat{\otimes}_{\pi} X_i \in \mathfrak{g}^{\circ} \widehat{\otimes}_{\pi} \mathfrak{g} \cong \text{End}(\mathfrak{g})$ that corresponds to $\text{Id} \in \text{End}(\mathfrak{g})$, we have $v \cdot f = \sum_{i \in I} f(X_i)v \cdot f^i$. Hence the claim follows from the fact that $v \mapsto X_i \otimes_{\pi} v \cdot f^i$ defines a (left) \mathfrak{g} -coaction. \square

Corollary 4.4. *Given a topological Lie algebra \mathfrak{g} , the category of topological left \mathfrak{g} -comodules, and the category of topological right $S(\mathfrak{g}^*)$ -modules are isomorphic.*

Next, a topological analogue of [27, Lemma 5.3] is in order.

Proposition 4.5. *Let V be a t.v.s., \mathfrak{g} a topological Lie algebra, and $U(\mathfrak{g})$ be its universal enveloping Hopf algebra. If V is a topological left $U(\mathfrak{g})$ -comodule, then it is a topological left \mathfrak{g} -comodule.*

Proof. It follows from [1, Lemma A.2.2] that the canonical projection $\pi : U(\mathfrak{g}) \longrightarrow \mathfrak{g}$, and similarly $\pi \otimes \text{Id} : U(\mathfrak{g}) \widehat{\otimes}_{\pi} V \longrightarrow \mathfrak{g} \widehat{\otimes}_{\pi} V$ is continuous. Thus the claim follows. \square

The next proposition is about the reverse direction which is the one that algebraic comodules fail to take. However, we shall need the following terminology.

Definition 4.6. *A t.v.s V is called fixed-bounded if for any fixed $v \in V$ there is a family of seminorms $\{q_\lambda \mid \lambda \in \Lambda\}$, that defines the topology of V , such that*

$$\sup_{\lambda \in \Lambda} q_\lambda(v) < \infty.$$

Example 4.7. Any normable space is fixed-bounded.

We are now ready to prove our main result in this section.

Proposition 4.8. *Let \mathfrak{g} be a topological Lie algebra, $U(\mathfrak{g})$ its universal enveloping Hopf algebra, and V be a fixed-bounded left \mathfrak{g} -comodule by $\nabla_{\mathfrak{g}} : V \longrightarrow \mathfrak{g} \hat{\otimes}_{\pi} V$, where $\nabla_{\mathfrak{g}}(v) := v_{[-1]} \hat{\otimes}_{\pi} v_{[0]}$. Then, V is a topological $U(\mathfrak{g})$ -comodule via*

$$\nabla_{\mathfrak{g}}^{\text{exp}} : V \longrightarrow U(\mathfrak{g}) \hat{\otimes}_{\pi} V, \quad v \mapsto \sum_{k=0}^{\infty} \frac{1}{k!} v_{[-k]} \dots v_{[-1]} \hat{\otimes}_{\pi} v_{[0]}. \quad (4.3)$$

Proof. Let $v \in V$, and $\{q_\lambda \mid \lambda \in \Lambda\}$ is the family given in Definition 4.6. We first show that the series (4.3) converges. To this end, we consider the sequence

$$(s_n)_{n \in \mathbb{N}}, \quad s_n := \sum_{k=0}^n \frac{1}{k!} v_{[-k]} \dots v_{[-1]} \hat{\otimes}_{\pi} v_{[0]} \quad (4.4)$$

of partial sums. For any continuous semi-norm ρ on $U(\mathfrak{g}) \hat{\otimes}_{\pi} V$, using [34, Prop. 7.7] and the continuity of the \mathfrak{g} -coaction, for any $n \geq m$ we have $\lambda_i \in \Lambda$ such that

$$\begin{aligned} \rho(s_n - s_m) &\leq \sum_{k=m+1}^n \frac{1}{k!} \rho(v_{[-k]} \dots v_{[-1]} \hat{\otimes}_{\pi} v_{[0]}) \\ &\leq \sum_{k=m+1}^n \frac{\mu_{\lambda_k}(v)}{k!} \leq \sup_{\lambda \in \Lambda} \mu_{\lambda}(v) \sum_{k=m+1}^n \frac{1}{k!}. \end{aligned}$$

Therefore, the sequence (4.4) of partial sums is Cauchy, and hence converges in $U(\mathfrak{g}) \hat{\otimes}_{\pi} V$. Finally, by [27, Prop. 5.7], (4.3) is indeed a coaction. \square

As a result, we can consider the following simple example as a \mathfrak{g} -comodule which is not locally conilpotent.

Example 4.9. Let $\mathfrak{g} = \langle X \rangle$ be a one dimensional trivial Lie algebra. Then $k \longrightarrow \mathfrak{g} \hat{\otimes}_\pi k$, $\mathbf{1} \mapsto X \hat{\otimes}_\pi \mathbf{1}$ determines a left \mathfrak{g} -coaction on k . As a result of Proposition 4.8 we have the exponentiation

$$k \longrightarrow U(\mathfrak{g}) \hat{\otimes}_\pi k, \quad \mathbf{1} \mapsto \exp(X) \hat{\otimes}_\pi \mathbf{1}.$$

On the next result, we extend the scope of Proposition 4.8 to $S(\mathfrak{g}^*)$, equipped with the Koszul coaction. We thus obtain an example of an infinite dimensional \mathfrak{g} -comodule which is not locally conilpotent.

To this end we will need the following terminology. A topological algebra is called locally multiplicatively convex, if its topology can be defined by a family of submultiplicative¹ semi-norms. In order to check whether an algebra is locally multiplicatively convex, one uses the following lemma of [18], see also [26, Coroll. 1.3].

Lemma 4.10. *Let A be a locally convex topological algebra. Suppose A has a base \mathcal{U} of absolutely convex² neighborhoods at 0 with the property that for each $V \in \mathcal{U}$ there exists $U \in \mathcal{U}$ and $C > 0$ such that $UV \subseteq CV$. Then A is locally multiplicatively convex.*

We now see that this is the case for $U(\mathfrak{g})$ and $S(\mathfrak{g}^*)$ with the natural topology.

Lemma 4.11. *For a topological Lie algebra \mathfrak{g} , the universal enveloping algebra $U(\mathfrak{g})$ is locally multiplicatively convex.*

Proof. The canonical filtration $(U_n(\mathfrak{g}))_{n \in \mathbb{N}}$, see for instance [6, Subsect. 2.3.1], determines a sequence of definition for the natural topology on $U(\mathfrak{g})$.

Let V be an absolutely convex neighbourhood of 0. Let also

$$U := \{\alpha \mathbf{1} \mid \alpha \in \mathbb{C}, |\alpha| \leq 1\} \subseteq U_0(\mathfrak{g}).$$

It follows immediately that $U \cap U_n(\mathfrak{g}) = U$ is convex for any $n \in \mathbb{N}$, and that U is balanced. Moreover, since V is balanced, $UV \subseteq V$, and hence the claim. \square

¹A semi-norm $\rho : A \longrightarrow [0, \infty)$ is said to be submultiplicative if $\rho(ab) \leq \rho(a)\rho(b)$ for any $a, b \in A$.

²Convex and balanced, where a subset \mathcal{U} of a l.c. t.v.s. W is called balanced if $\lambda w \in \mathcal{U}$ for any $w \in W$ and any $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$.

The same is true for the symmetric algebra $S(\mathfrak{g}^*)$.

Lemma 4.12. *For a topological Lie algebra \mathfrak{g} , the symmetric algebra $S(\mathfrak{g}^*)$ is locally multiplicatively convex.*

Proposition 4.13. *Let \mathfrak{g} be a finite dimensional Lie algebra. Then the symmetric algebra $S(\mathfrak{g}^*)$ equipped with the Koszul coaction (4.1) is a topological $U(\mathfrak{g})$ -comodule via (4.3).*

Proof. Recalling the element $\sum_{i \in I} f^i \hat{\otimes}_\pi X_i \in \mathfrak{g}^\circ \hat{\otimes}_\pi \mathfrak{g} \cong \text{End}(\mathfrak{g})$, we may express the Koszul coaction (4.1) as $\nabla_{\mathfrak{g}}(R) = \sum_{i \in I} X_i \hat{\otimes}_\pi R f^i$ for any $R \in S(\mathfrak{g}^*)$. As a result we have

$$\nabla_{\mathfrak{g}}^{\text{exp}}(R) = \sum_{k=0}^{\infty} \frac{1}{k!} X_{i_k} \dots X_{i_1} \hat{\otimes}_\pi R f^{i_1} \dots f^{i_k}.$$

Then, along the lines of the proof of Proposition 4.8, for

$$s_n := \sum_{k=0}^n \frac{1}{k!} X_{i_k} \dots X_{i_1} \hat{\otimes}_\pi R f^{i_1} \dots f^{i_k}, \quad (4.5)$$

and any seminorm $\rho_{p,q}$ on $U(\mathfrak{g}) \hat{\otimes}_\pi S(\mathfrak{g}^*)$ we observe that

$$\begin{aligned} \rho_{p,q}(s_n - s_m) &\leq \sum_{k=m+1}^n \frac{1}{k!} p(X_{i_k} \dots X_{i_1}) q(R f^{i_1} \dots f^{i_k}) \\ &\leq \sum_{k=m+1}^n \frac{1}{k!} p(X_{i_k}) \dots p(X_{i_1}) q(R) q(f^{i_1}) \dots q(f^{i_k}), \end{aligned}$$

since $U(\mathfrak{g})$ and $S(\mathfrak{g}^*)$ are locally multiplicatively convex by Lemma 4.11 and Lemma 4.12. It then follows from [34, Prop. 7.7] that there is a semi-norm μ on $S(\mathfrak{g}^*)$ such that

$$\sum_{i_j} p(X_{i_j}) q(f^{i_j}) \leq \mu(1).$$

As a result,

$$\rho_{p,q}(s_n - s_m) \leq \sum_{k=m+1}^n \frac{1}{k!} \mu(1)^k q(R).$$

Therefore, the sequence $(s_n)_{n \in \mathbb{N}}$ of partial sums given by (4.5) is Cauchy, and hence converges in $U(\mathfrak{g}) \hat{\otimes}_\pi S(\mathfrak{g}^*)$. \square

We conclude this subsection with a short discussion on AYD modules over topological Lie algebras.

Definition 4.14. *Let V be a topological right module / left comodule over a topological Lie algebra \mathfrak{g} . We call V a right-left topological AYD module over \mathfrak{g} if*

$$\nabla(v \cdot X) = v_{[-1]} \widehat{\otimes}_\pi v_{[0]} \cdot X + [v_{[-1]}, X] \widehat{\otimes}_\pi v_{[0]}. \quad (4.6)$$

In addition, V is called stable if

$$v_{[0]} \cdot v_{[-1]} = 0. \quad (4.7)$$

Proposition 4.15. *Let \mathfrak{g} be a topological Lie algebra, and V a topological \mathfrak{g} -module / comodule. Then V is a topological right-left AYD module over \mathfrak{g} if and only if it is a topological right-left AYD module over $U(\mathfrak{g})$.*

Proof. It follows from Proposition 4.5 that if V is a left $U(\mathfrak{g})$ -comodule, then it is a left \mathfrak{g} -comodule. On the next step we use Proposition 4.8 to conclude that V is a left $U(\mathfrak{g})$ -comodule if it is a \mathfrak{g} -comodule. Finally, the AYD condition is similar to [27, Lemma 5.10]. \square

4.2 Cyclic cohomology theories for topological Lie algebras

In this subsection we extend to topological Lie algebras, the two cyclic complexes introduced in [27]. More precisely, the complex associated to a Lie algebra and a SAYD module over it, generalizing the Lie algebra homology complex, and the one associated to a Lie algebra and a unimodular SAYD module over it, generalizing the Lie algebra cohomology complex.

We shall begin with the unimodular stability [27, Prop. 2.4] on the level of topological Lie algebras.

Definition 4.16. *Let \mathfrak{g} be a topological Lie algebra, and V a topological right \mathfrak{g} -module / left \mathfrak{g} -comodule. Then V is called unimodular stable over \mathfrak{g} if any element of V is annihilated by $\sum_{i \in I} X_i \widehat{\otimes}_\pi f^i \in \mathfrak{g} \widehat{\otimes}_\pi \mathfrak{g}^\circ \cong \text{End}(\mathfrak{g})$, that is,*

$$v \cdot \left(\sum_{i \in I} X_i \widehat{\otimes}_\pi f^i \right) = 0. \quad (4.8)$$

If \mathfrak{g} is finite dimensional, then $\sum_{i \in I} X_i \widehat{\otimes}_\pi f^i \in \mathfrak{g}^* \otimes \mathfrak{g}$ consists of a dual pair of bases, and the above definition coincides with the one given in [27]. Similarly, the stability can be given by the trivial action of $\sum_{i \in I} f^i \widehat{\otimes}_\pi X_i \in \mathfrak{g}^\circ \widehat{\otimes}_\pi \mathfrak{g}$.

Let us next recall the cohomology of the topological Lie algebras from [25, Def. I.1], see also [35]. Given a topological Lie algebra \mathfrak{g} , and a topological \mathfrak{g} -module V , let $W_{\text{top}}^n(\mathfrak{g}, V)$ denote the space of continuous alternating maps $\mathfrak{g}^{\times n} \rightarrow V$ for $n \geq 0$. In other words, $W_{\text{top}}^n(\mathfrak{g}, V)$ is the set of continuous n -cochains with values in V . Then, $W_{\text{top}}^n(\mathfrak{g}, V)$ is a differential complex with the differential

$$\begin{aligned} d_{\text{CE}} : W_{\text{top}}^n(\mathfrak{g}, V) &\longrightarrow W_{\text{top}}^{n+1}(\mathfrak{g}, V), \\ d_{\text{CE}}(\alpha)(Y_0, \dots, Y_n) &:= \sum_{i < j} (-1)^{i+j} \alpha([Y_i, Y_j], Y_0, \dots, \widehat{Y}_i, \dots, \widehat{Y}_j, \dots, Y_n) \\ &+ \sum_{j=0}^n (-1)^{j+1} \alpha(Y_0, \dots, \widehat{Y}_j, \dots, Y_n) \cdot Y_j. \end{aligned}$$

Similarly we define the Koszul boundary map, see [27, Sect. 3.2], by

$$\begin{aligned} d_{\text{K}} : W_{\text{top}}^{n+1}(\mathfrak{g}, V) &\longrightarrow W_{\text{top}}^n(\mathfrak{g}, V), \\ d_{\text{K}}(\beta)(Y_1, \dots, Y_n) &:= \sum_{i \in I} \iota_{X_i}(\beta)(Y_1, \dots, Y_n) \cdot f^i. \end{aligned}$$

Since the truncation and the $S(\mathfrak{g}^*)$ -action are continuous, by Proposition 4.3, the Koszul boundary map restricts to the continuous cochains. Moreover, for any $\beta \in W_{\text{top}}^{n+1}(\mathfrak{g}, V)$,

$$d_{\text{K}}^2(\beta)(Y_1, \dots, Y_{n-1}) = \sum_{i, j \in I} \iota_{X_i} \iota_{X_j}(\beta)(Y_1, \dots, Y_{n-1}) \cdot f^j f^i = 0,$$

that is, $d_{\text{K}}^2 = 0$.

We are now ready to define a cyclic cohomology theory for topological Lie algebras. To this end we record the following analogue of [27, Thm. 2.4].

Proposition 4.17. *Given a topological Lie algebra \mathfrak{g} , and a topological (right) \mathfrak{g} -module / (left) \mathfrak{g} -comodule V ,*

$$(W_{\text{top}}^*(\mathfrak{g}, V), d_{\text{CE}} + d_{\text{K}}), \quad W_{\text{top}}^*(\mathfrak{g}, V) := \bigoplus_{n \geq 0} W_{\text{top}}^n(\mathfrak{g}, V) \quad (4.9)$$

is a differential complex if and only if V is a unimodular SAYD module over the Lie algebra \mathfrak{g} .

Proof. Given any $\gamma \in W_{\text{top}}^n(\mathfrak{g}, V)$ for $n \geq 0$, we have

$$\begin{aligned}
d_{\text{CE}}d_K(\gamma)(Y_0, \dots, Y_{n-1}) &= \\
&\sum_{i < j} (-1)^{i+j} d_K(\gamma)([Y_i, Y_j], Y_0, \dots, \widehat{Y}_i, \dots, \widehat{Y}_j, \dots, Y_{n-1}) \\
&+ \sum_{j=0}^{n-1} (-1)^{j+1} d_K(\gamma)(Y_0, \dots, \widehat{Y}_j, \dots, Y_{n-1}) \cdot Y_j \\
&= \sum_{k \in I} \sum_{i < j} (-1)^{i+j} \gamma(X_k, [Y_i, Y_j], Y_0, \dots, \widehat{Y}_i, \dots, \widehat{Y}_j, \dots, Y_{n-1}) \cdot f^k \\
&+ \sum_{k \in I} \sum_{j=0}^{n-1} (-1)^{j+1} (\gamma(X_k, Y_0, \dots, \widehat{Y}_j, \dots, Y_{n-1}) \cdot f^k) \cdot Y_j,
\end{aligned}$$

as well as

$$\begin{aligned}
d_K d_{\text{CE}}(\gamma)(Y_0, \dots, Y_{n-1}) &= \sum_{k \in I} d_{\text{CE}}(\gamma)(X_k, Y_0, \dots, Y_{n-1}) \cdot f^k \\
&= \sum_{k \in I} \sum_{i < j} (-1)^{i+j} \gamma([Y_i, Y_j], X_k, Y_0, \dots, \widehat{Y}_i, \dots, \widehat{Y}_j, \dots, Y_{n-1}) \cdot f^k \\
&+ \sum_{k \in I} \sum_{j=0}^{n-1} (-1)^{j+1} \gamma([X_k, Y_j], Y_0, \dots, \widehat{Y}_j, \dots, Y_{n-1}) \cdot f^k \\
&- \sum_{k \in I} (\gamma(Y_0, \dots, Y_{n-1}) \cdot X_k) \cdot f^k \\
&+ \sum_{k \in I} \sum_{j=0}^{n-1} (-1)^j (\gamma(X_k, Y_0, \dots, \widehat{Y}_j, \dots, Y_{n-1}) \cdot Y_j) \cdot f^k.
\end{aligned}$$

As a result, (4.9) is a differential complex, *i.e.* $d_{\text{CE}}d_K + d_K d_{\text{CE}} = 0$ if and only if for any $\gamma \in W_{\text{top}}^n(\mathfrak{g}, V)$, and any $Y_0, \dots, Y_{n-1} \in \mathfrak{g}$,

$$\begin{aligned}
&\sum_{k \in I} \sum_{j=0}^{n-1} (-1)^{j+1} [(\gamma(X_k, Y_0, \dots, \widehat{Y}_j, \dots, Y_{n-1}) \cdot f^k) \cdot Y_j \\
&\quad - \sum_{k \in I} (\gamma(X_k, Y_0, \dots, \widehat{Y}_j, \dots, Y_{n-1}) \cdot Y_j) \cdot f^k] \\
&+ \sum_{k \in I} \sum_{j=0}^{n-1} (-1)^{j+1} \gamma([X_k, Y_j], Y_0, \dots, \widehat{Y}_j, \dots, Y_{n-1}) \cdot f^k \\
&- \sum_{k \in I} (\gamma(Y_0, \dots, Y_{n-1}) \cdot X_k) \cdot f^k = 0.
\end{aligned}$$

This last equality for a $\gamma \in W_{\text{top}}^0(\mathfrak{g}, V) := V$ is

$$\sum_{k \in I} (\gamma \cdot X_k) \cdot f^k = 0,$$

that is, the unimodular stability. For a $\gamma \in W_{\text{top}}^1(\mathfrak{g}, V)$, on the other hand, we obtain

$$\begin{aligned} \sum_{k \in I} (\gamma(X_k) \cdot Y) \cdot f^k - \sum_{k \in I} (\gamma(X_k) \cdot f^k) \cdot Y \\ - \sum_{k \in I} \gamma([X_k, Y]) \cdot f^k - \sum_{k \in I} (\gamma(Y) \cdot X_k) \cdot f^k = \\ \sum_{k \in I} (\gamma(X_k) \cdot Y) \cdot f^k - \sum_{k \in I} (\gamma(X_k) \cdot f^k) \cdot Y - \sum_{k \in I} \gamma([X_k, Y]) \cdot f^k = 0, \end{aligned}$$

which is precisely the AYD compatibility. \square

Definition 4.18. *The cyclic cohomology of \mathfrak{g} with coefficients in a unimodular SAYD module V over \mathfrak{g} , which is denoted by $\widetilde{HC}_{\text{top}}^*(\mathfrak{g}, V)$, is defined to be the total cohomology of the bicomplex*

$$W_{\text{top}}^{p,q}(\mathfrak{g}, V) = \begin{cases} W_{\text{top}}^{q-p}(\mathfrak{g}, V), & \text{if } 0 \leq p \leq q, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, the periodic cyclic cohomology of \mathfrak{g} with coefficients in a unimodular SAYD module V over \mathfrak{g} , which is denoted by $\widetilde{HP}_{\text{top}}^(\mathfrak{g}, V)$, is defined to be the total cohomology of*

$$W_{\text{top}}^{p,q}(\mathfrak{g}, V) = \begin{cases} W_{\text{top}}^{q-p}(\mathfrak{g}, V), & \text{if } p \leq q, \\ 0, & \text{otherwise.} \end{cases}$$

Let \mathfrak{g} be a topological Lie algebra, and V a topological right \mathfrak{g} -module / left \mathfrak{g} -comodule. Similar to [27, Sect. 4.2], we can associate a cyclic cohomology theory to a topological Lie algebra \mathfrak{g} , and a SAYD module over it, by setting

$$C_n^{\text{top}}(\mathfrak{g}, V) := \wedge^n \mathfrak{g} \hat{\otimes}_{\pi} V$$

with two continuous differentials

$$\partial_{\text{CE}} : C_{n+1}^{\text{top}}(\mathfrak{g}, V) \longrightarrow C_n^{\text{top}}(\mathfrak{g}, V),$$

$$\partial_{\text{CE}}(Y_0 \wedge \cdots \wedge Y_n \widehat{\otimes}_{\pi} v) = \sum_{j=0}^n (-1)^{j+1} Y_0 \wedge \cdots \wedge \widehat{Y}_j \wedge \cdots \wedge Y_n \widehat{\otimes}_{\pi} v \cdot Y_j +$$

$$\sum_{j,k=0}^n (-1)^{j+k} [Y_j, Y_k] \wedge Y_0 \wedge \cdots \wedge \widehat{Y}_j \wedge \cdots \wedge \widehat{Y}_k \wedge \cdots \wedge Y_n \widehat{\otimes}_{\pi} v,$$

and

$$\partial_{\text{K}} : C_n^{\text{top}}(\mathfrak{g}, V) \longrightarrow C_{n+1}^{\text{top}}(\mathfrak{g}, V),$$

$$\partial_{\text{K}}(Y_1 \wedge \cdots \wedge Y_n \widehat{\otimes}_{\pi} v) = v_{[-1]} \wedge Y_1 \wedge \cdots \wedge Y_n \widehat{\otimes}_{\pi} v_{[0]}.$$

Having defined the (co)boundary maps, we can immediately state the following analogue of [27, Prop. 4.2].

Proposition 4.19. *Let \mathfrak{g} be a topological Lie algebra, and V a topological right \mathfrak{g} -module / left \mathfrak{g} -comodule. Then $(C_*^{\text{top}}(\mathfrak{g}, V), \partial_{\text{CE}} + \partial_{\text{K}})$ is a differential complex if and only if V is a SAYD module over \mathfrak{g} .*

As a result, we define the cyclic cohomology of a topological Lie algebra, and a topological SAYD module associated to it.

Definition 4.20. *The cyclic cohomology of \mathfrak{g} with coefficients in a SAYD module V over \mathfrak{g} , which is denoted by $HC_{\text{top}}^*(\mathfrak{g}, V)$, is defined to be the total cohomology of the bicomplex*

$$C_{p,q}^{\text{top}}(\mathfrak{g}, V) = \begin{cases} C_{q-p}^{\text{top}}(\mathfrak{g}, V), & \text{if } 0 \leq p \leq q, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, the periodic cyclic cohomology of \mathfrak{g} with coefficients in a SAYD module V over \mathfrak{g} , which is denoted by $HP_{\text{top}}^(\mathfrak{g}, V)$, is defined to be the total cohomology of*

$$C_{p,q}^{\text{top}}(\mathfrak{g}, V) = \begin{cases} C_{q-p}^{\text{top}}(\mathfrak{g}, V), & \text{if } p \leq q, \\ 0, & \text{otherwise.} \end{cases}$$

5 Computation

In this section we compute the Hopf cyclic cohomology of topological Lie-Hopf algebras. To this end, we consider in the first subsection the coalgebra Hochschild cohomology of $\mathcal{F}^\infty(G)$, which appears in the E_1 -page of a spectral sequence computing the Hopf-cyclic cohomology of the topological Lie-Hopf algebra $\mathcal{F}^\infty(G_2) \bowtie_{\pi} U(\mathfrak{g}_1)$ associated to a matched pair (G_1, G_2) of Lie groups.

5.1 Coalgebra Hochschild cohomology of $\mathcal{F}^\infty(G)$

In this subsection we identify the coalgebra Hochschild cohomology of the Hopf algebra $\mathcal{F}^\infty(G)$ of differentiable functions on a real analytic group G , with coefficients in a differentiable G -module V , with the differentiable cohomology (and hence the continuous cohomology [11]) of the group G . The latter, in turn, is identified with the relative Lie algebra cohomology of the Lie algebra \mathfrak{g} of G , relative to the Lie algebra of a maximal compact subgroup of G , [36, 7, 11].

Let us first recall the differentiable cohomology from [11]. For $n \geq 0$, let $C_d^n(G, V)$ be the space of all differentiable maps $G^{\times n} \rightarrow V$,

$$C_d^*(G, V) = \bigoplus_{n \geq 0} C_d^n(G, V),$$

and let

$$\begin{aligned} d(c)(g_1, \dots, g_{n+1}) &:= g_1 \cdot c(g_2, \dots, g_{n+1}) + \\ &\sum_{i=1}^n (-1)^i c(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) + (-1)^{n+1} c(g_1, \dots, g_n). \end{aligned} \tag{5.1}$$

Then $(C_d^*(G, V), d)$ is a differential graded complex whose cohomology is called the differentiable cohomology of G with coefficients in the differentiable G -module V , and is denoted by $H_d^*(G, V)$.

Proposition 5.1. *Let V be a left $\mathcal{F}^\infty(G)$ -comodule via $v \mapsto v^{<-1>} \widehat{\otimes}_\pi v^{<0>}$. Then the map*

$$\begin{aligned} \Psi : C^n(\mathcal{F}^\infty(G), V) &\longrightarrow C_d^n(G, V), \\ \Psi(v \widehat{\otimes}_\pi f^1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi f^n)(g_1, \dots, g_n) &:= v f^1(g_n^{-1}) f^2(g_{n-1}^{-1}) \dots f^n(g_1^{-1}) \end{aligned}$$

is an isomorphism of complexes.

Proof. We note by Proposition 2.8 that being a left $\mathcal{F}^\infty(G)$ -comodule, V is a differentiable right G -module. Thus, V possesses the left G -module structure given by $x \cdot v := v \cdot x^{-1}$. Accordingly we have

$$\begin{aligned}
& d\left(\Psi(v \widehat{\otimes}_\pi f^1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi f^n)\right)(g_1, \dots, g_{n+1}) = \\
& \quad g_1 \cdot \Psi(v \widehat{\otimes}_\pi f^1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi f^n)(g_2, \dots, g_{n+1}) + \\
& \quad \sum_{j=1}^n (-1)^j \Psi(v \widehat{\otimes}_\pi f^1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi f^n)(g_1, \dots, g_j g_{j+1}, \dots, g_{n+1}) + \\
& \quad \Psi(v \widehat{\otimes}_\pi f^1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi f^n)(g_1, \dots, g_n) = \\
& \quad g_1 \cdot v f^1(g_{n+1}^{-1}) \dots f^n(g_2^{-1}) + \\
& \quad \sum_{j=1}^n (-1)^j v f^1(g_{n+1}^{-1}) \dots f^{n-j+1}((g_j g_{j+1})^{-1}) \cdot f^n(g_1^{-1}) + \\
& \quad (-1)^{n+1} v f^1(g_n^{-1}) \cdot f^n(g_1^{-1}), \tag{5.2}
\end{aligned}$$

and

$$\begin{aligned}
& \Psi(b(v \widehat{\otimes}_\pi f^1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi f^n))(g_1, \dots, g_{n+1}) = \\
& \Psi(v \widehat{\otimes}_\pi \varepsilon \widehat{\otimes}_\pi f^1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi f^n)(g_1, \dots, g_{n+1}) + \\
& \quad \sum_{j=1}^n (-1)^j \Psi(v \widehat{\otimes}_\pi f^1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi \Delta(f^j) \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi f^n)(g_1, \dots, g_{n+1}) + \\
& \quad (-1)^{n+1} \Psi(v \widehat{\otimes}_\pi f^1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi f^n \widehat{\otimes}_\pi v \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi f^n)(g_1, \dots, g_{n+1}) = \\
& \quad v f^1(g_n^{-1}) \dots f^n(g_1^{-1}) + \\
& \quad \sum_{j=1}^n (-1)^j v f^1(g_{n+1}^{-1}) \dots f^j(g_{n-j+2}^{-1} g_{n-j+1}^{-1}) \dots f^n(g_1^{-1}) + \\
& \quad (-1)^{n+1} g_1 \cdot v f^1(g_{n+1}^{-1}) \dots f^n(g_1^{-1}).
\end{aligned}$$

On the other hand, for $1 \leq j \leq n$, one has $1 \leq k := n+1-j \leq n$, and by this substitution we rewrite (5.2) as

$$\begin{aligned}
& d\left(\Psi(v \widehat{\otimes}_\pi f^1 \widehat{\otimes}_\pi \dots \widehat{\otimes}_\pi f^n)\right)(g_1, \dots, g_{n+1}) = \\
& \quad g_1 \cdot v f^1(g_{n+1}^{-1}) \dots f^n(g_2^{-1}) + \\
& \quad \sum_{k=1}^n (-1)^{n+1-k} v f^1(g_{n+1}^{-1}) \dots f^k((g_{n+1-k} g_{n+2-k})^{-1}) \cdot f^n(g_1^{-1}) + \\
& \quad (-1)^{n+1} v f^1(g_n^{-1}) \cdot f^n(g_1^{-1}),
\end{aligned}$$

As a result, we see that

$$d(\Psi(v \hat{\otimes}_\pi f^1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi f^n)) = (-1)^{n+1} \Psi(b(v \hat{\otimes}_\pi f^1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi f^n)).$$

□

Following [23, 11] we denote by $H_c^*(G, V)$ the continuous cohomology of a topological group G , with coefficients in a continuous (left) G -module. A continuous G -module is defined to be a Hausdorff t.v.s. such that the G -action $G \times V \rightarrow V$ is continuous, and $H_c^*(G, V)$ is defined to be the homology with respect to the coboundary (5.1).

Let us next recall the relation between the continuous cohomology and the differentiable cohomology from [11, Thm. 5.1]. To this end, we need to recall the integrability of the coefficient space [11], see also [23, 2.13].

Definition 5.2. *Let V be a t.v.s., and G a locally compact topological group. Let also $F(G, V)$ be the space of all continuous maps $G \rightarrow V$ with compact support, topologized by the compact-open topology. Then V is called G -integrable if there is a continuous map $J_G : F(G, V) \rightarrow V$ such that for any $\gamma \in V^\circ$ and any $f \in F(G, V)$,*

$$\gamma(J_G(f)) = I_G(\gamma \circ f), \quad (5.3)$$

where I_G is a Haar integral on G .

In case G is replaced by \mathbb{R} , and the Haar integral by an ordinary integral over $[0, 1]$, V is called $[0, 1]$ -integrable. The following is [11, Thm. 5.1].

Theorem 5.3. *Let G be a real analytic group, and V a differentiable (left) G -module. If V is locally convex and G -integrable, then the canonical homomorphism $H_d^*(G, V) \rightarrow H_c^*(G, V)$ is an isomorphism.*

We note from [23, 2.13(iii)] that any finite dimensional Hausdorff, or more generally any continuous G -module which is complete as a t.v.s. is G -integrable, see [11, Sect. 6].

The passage to the continuous group cohomology enables us to link the Hochschild cohomology of $\mathcal{F}^\infty(G)$ to the (relative) Lie algebra cohomology, [11, Sect. 6].

Theorem 5.4. *Let G be a real analytic group, and K be a maximal compact subgroup of G . Let also V be a locally convex G -, K -, and $[0, 1]$ -integrable differentiable G -module. Then,*

$$H_c^*(G, V) \cong H_{CE}^*(\mathfrak{g}, \mathfrak{k}, V). \quad (5.4)$$

5.2 Hopf-cyclic cohomology of $\mathcal{F}^\infty(G_2) \hat{\bowtie}_\pi U(\mathfrak{g}_1)$

In this subsection we shall identify the (periodic) Hopf-cyclic cohomology of the Hopf algebra $\mathcal{F}^\infty(G_2) \hat{\bowtie}_\pi U(\mathfrak{g}_1)$ of Example 2.15 with the Lie algebra cohomology of $\mathfrak{g}_1 \bowtie \mathfrak{g}_2$ relative to \mathfrak{k} , the maximal compact subalgebra of \mathfrak{g}_2 , via a van Est type isomorphism. For the construction of a MPI (δ, σ) over a bicrossed product Hopf algebra, we refer the reader to [29, Thm. 3.2].

Theorem 5.5. *Let (G_1, G_2) be a matched pair of Lie groups, with Lie algebras $(\mathfrak{g}_1, \mathfrak{g}_2)$. Then the periodic Hopf-cyclic cohomology of the Hopf algebra $\mathcal{F}^\infty(G_2) \hat{\bowtie}_\pi U(\mathfrak{g}_1)$ with coefficients in the canonical MPI ${}^\sigma k_\delta$ is isomorphic with the Lie algebra cohomology, with trivial coefficients, of $\mathfrak{g}_1 \bowtie \mathfrak{g}_2$ relative to the maximal compact Lie subalgebra \mathfrak{k} of \mathfrak{g}_2 . In short,*

$$HP_{top}^*(\mathcal{F}^\infty(G_2) \hat{\bowtie}_\pi U(\mathfrak{g}_1), {}^\sigma k_\delta) \cong \widetilde{HP}^*(\mathfrak{g}_1 \bowtie \mathfrak{g}_2, \mathfrak{k}).$$

Proof. By [19, Prop. 3.16], see also [29, Prop. 5.1] and [28, Thm. 4.6], Hopf-cyclic cohomology of the bicrossed product Hopf algebra $\mathcal{F}^\infty(G_2) \hat{\bowtie}_\pi U(\mathfrak{g}_1)$ is computed by the bicomplex

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \uparrow d_{CE} & & \uparrow d_{CE} & & \uparrow d_{CE} & \\
 \wedge^2 \mathfrak{g}_1^* & \xrightarrow{b} & \wedge^2 \mathfrak{g}_1^* \hat{\otimes}_\pi \mathcal{F}^\infty(G_2) & \xrightarrow{b} & \wedge^2 \mathfrak{g}_1^* \hat{\otimes}_\pi \mathcal{F}^\infty(G_2)^{\otimes 2} & \xrightarrow{b} & \dots \\
 \uparrow d_{CE} & & \uparrow d_{CE} & & \uparrow d_{CE} & & \\
 \mathfrak{g}_1^* & \xrightarrow{b} & \mathfrak{g}_1^* \hat{\otimes}_\pi \mathcal{F}^\infty(G_2) & \xrightarrow{b} & \mathfrak{g}_1^* \hat{\otimes}_\pi \mathcal{F}^\infty(G_2)^{\otimes 2} & \xrightarrow{b} & \dots \\
 \uparrow d_{CE} & & \uparrow d_{CE} & & \uparrow d_{CE} & & \\
 k & \xrightarrow{b} & \mathcal{F}^\infty(G_2) & \xrightarrow{b} & \mathcal{F}^\infty(G_2)^{\otimes 2} & \xrightarrow{b} & \dots
 \end{array} , \tag{5.5}$$

where $d_{CE} : \wedge^q \mathfrak{g}_1^* \hat{\otimes}_\pi \mathcal{F}^\infty(G_2)^{\otimes p} \longrightarrow \wedge^{q+1} \mathfrak{g}_1^* \hat{\otimes}_\pi \mathcal{F}^\infty(G_2)^{\otimes p}$ is the Lie algebra cohomology of \mathfrak{g}_1 , with coefficients in $\mathcal{F}^\infty(G_2)^{\otimes p}$. Similarly, b^* is the coalgebra Hochschild cohomology coboundary with coefficients in the $\mathcal{F}^\infty(G_2)$ -comodule $\wedge^q \mathfrak{g}_1^*$. Since $\wedge^q \mathfrak{g}_1^*$ is a differentiable G_2 -module, by Proposition 2.8 it is an $\mathcal{F}^\infty(G_2)$ -comodule. As a result, by

Proposition 5.1 the bicomplex (5.5) is isomorphic with the bicomplex

$$\begin{array}{ccccccc}
& \vdots & & \vdots & & \vdots & \\
& \uparrow d_{\text{CE}} & & \uparrow d_{\text{CE}} & & \uparrow d_{\text{CE}} & \\
C_d^0(G_2, \wedge^2 \mathfrak{g}_1^*) & \xrightarrow{d^*} & C_d^1(G_2, \wedge^2 \mathfrak{g}_1^*) & \xrightarrow{d^*} & C_d^2(G_2, \wedge^2 \mathfrak{g}_1^*) & \xrightarrow{d^*} & \dots \\
& \uparrow d_{\text{CE}} & & \uparrow d_{\text{CE}} & & \uparrow d_{\text{CE}} & \\
C_d^0(G_2, \mathfrak{g}_1^*) & \xrightarrow{d^*} & C_d^1(G_2, \mathfrak{g}_1^*) & \xrightarrow{d^*} & C_d^2(G_2, \mathfrak{g}_1^*) & \xrightarrow{d^*} & \dots \\
& \uparrow d_{\text{CE}} & & \uparrow d_{\text{CE}} & & \uparrow d_{\text{CE}} & \\
C_d^0(G_2, k) & \xrightarrow{d^*} & C_d^1(G_2, k) & \xrightarrow{d^*} & C_d^2(G_2, k) & \xrightarrow{d^*} & \dots
\end{array},$$

where $d^* : C_d^p(G_2, \wedge^q \mathfrak{g}_1^*) \rightarrow C_d^{p+1}(G_2, \wedge^q \mathfrak{g}_1^*)$ is the differentiable cohomology of G_2 , with coefficients in $\wedge^q \mathfrak{g}_1^*$.

By [11, Thm. 6.1], see also [7, 36], one has the map

$$\mathcal{F}^{p,q} : C_{\text{CE}}^p(\mathfrak{g}_2, \mathfrak{k}, \wedge^p \mathfrak{g}_1^*) \rightarrow C_d^p(G_2, \wedge^p \mathfrak{g}_1^*)$$

of bicomplexes [20], which induces an isomorphism on the level of total cohomologies, as well as an isomorphism on the level of row cohomologies. \square

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